

THE QUANTUM BLACK-SCHOLES EQUATION

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ABSTRACT. Motivated by the work of Segal and Segal in [16] on the Black-Scholes pricing formula in the quantum context, we study a quantum extension of the Black-Scholes equation within the context of Hudson-Parthasarathy quantum stochastic calculus,. Our model includes stock markets described by quantum Brownian motion and Poisson process.

1. THE MERTON-BLACK-SCHOLES OPTION PRICING MODEL

An *option* is a ticket which is bought at time $t = 0$ and which allows the buyer at (in the case of *European call* options) or until (in the case of *American call* options) time $t = T$ (the *time of maturity* of the option) to buy a share of stock at a fixed *exercise price* K . In what follows we restrict to European call options. The question is: how much should one be willing to pay to buy such an option? Let X_T be a *reasonable price*. According to the definition given by Merton, Black, and Scholes (M-B-S) an investment of this reasonable price in a mixed *portfolio* (i.e part is invested in stock and part in bond) at time $t = 0$, should allow the investor through a *self-financing strategy* (i.e one where the only change in the investor's wealth comes from changes of the prices of the stock and bond) to end up at time $t = T$ with an amount of $(X_T - K)^+ := \max(0, X_T - K)$ which is the same as the payoff, had the option been purchased (cf. [12]). Moreover, such a *reasonable* price allows for no *arbitrage* i.e, it does not allow for risk free unbounded profits. We assume that there are no *transaction costs* and that the portfolio is not made smaller by *consumption*. If $(a_t, b_t), t \in [0, T]$ is a self-financing *trading strategy* (i.e an amount a_t is invested in stock at time t and an amount b_t is invested in bond at the same time) then the *value* of the portfolio at time t is given by $V_t = a_t X_t + b_t \beta_t$ where, by the self-financing assumption, $dV_t = a_t dX_t + b_t d\beta_t$. Here X_t and β_t denote, respectively, the price of the stock and bond at time t . We assume that $dX_t = c X_t dt + \sigma X_t dB_t$ and $d\beta_t = \beta_t r dt$ where B_t is classical Brownian motion, $r > 0$ is the constant interest rate of the bond, $c > 0$ is the *mean rate of return*, and $\sigma > 0$ is the *volatility* of the stock. The assets a_t and b_t are in general stochastic processes. Letting $V_t = u(T - t, X_t)$ where $V_T = u(0, X_T) = (X_T - K)^+$ it can be shown (cf. [12]) that $u(t, x)$ is the solution of the Black-Scholes equation

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$$\begin{aligned}\frac{\partial}{\partial t}u(t, x) &= (0.5 \sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r x \frac{\partial}{\partial x} - r) u(t, x) \\ u(0, x) &= (X_T - K)^+, \quad x > 0, t \in [0, T]\end{aligned}$$

and it is explicitly given by

$$u(t, x) = x \Phi(g(t, x)) - K e^{-rt} \Phi(h(t, x))$$

where

$$g(t, x) = (\ln(x/K) + (r + 0.5 \sigma^2) t)(\sigma \sqrt{t})^{-1}, \quad h(t, x) = g(t, x) - \sigma \sqrt{t}$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!} \frac{x^{2n+1}}{2n+1}.$$

Thus a reasonable (in the sense described above) price for a European call option is

$$V_0 = u(T, X_0) = X_0 \Phi(g(T, X_0)) - K e^{-rT} \Phi(h(T, X_0))$$

and the self-financing strategy $(a_t, b_t), t \in [0, T]$ is given by

$$a_t = \frac{\partial}{\partial x} u(T-t, X_t), \quad b_t = \frac{u(T-t, X_t) - a_t X_t}{\beta_t}.$$

2. QUANTUM EXTENSION OF THE M-B-S MODEL

In recent years the fields of Quantum Economics and Quantum Finance have appeared in order to interpret erratic stock market behavior with the use of quantum mechanical concepts (cf. [3], [4],[6]-[9], [11], and [14]-[16]). While no approach has yet been proved prevalent, in [16] Segal and Segal introduced quantum effects into the Merton-Black-Scholes model in order to incorporate market features such as the impossibility of simultaneous measurement of prices and their instantaneous derivatives. They did that by adding to the Brownian motion B_t used to represent the evolution of public information affecting the market, a process Y_t which represents the influence of factors not simultaneously measurable with those involved in B_t . They then sketched a calculus for dealing with such processes. Segal and Segal concluded that the combined process $a B_t + b Y_t$ may be represented as (in their notation) $\Phi((a + ib) \chi_{[0,t]})$ where for a Hilbert space element f , $e^{i\Phi(f)}$ is the corresponding Weyl operator, and $\chi_{[0,t]}$ is the characteristic function of the interval $[0, t]$. In the context of the Hudson-Parthasarathy quantum stochastic calculus of [10] and [13] (see Theorem 20.10 of [13]) simple linear combinations of $\Phi(f)$ and $\Phi(if)$ define the Boson Fock space annihilator

and creator operators A_f and A_f^\dagger . Segal and Segal used $\Phi(\chi_{[0,t]})$ as the basic integrator process with integrands restricted to a special class of exponential processes. In view of the above reduction of Φ to A and A^\dagger , it makes sense to study option pricing using as integrators the annihilator and creator processes of Hudson-Parthasarathy quantum stochastic calculus, thus exploiting its much larger class of integrable processes than the one considered in [16]. The Hudson-Parthasarathy calculus has a wide range of applications. For applications to, for example, control theory we refer to [2], [5] and the references therein. Quantum stochastic calculus was designed to describe the dynamics of quantum processes and we propose that we use it to study the non commutative Merton-Black-Scholes model in the following formulation (notice that our model includes also the Poisson process): We replace (see [1] for details on quantization) the stock process $\{X_t / t \geq 0\}$ of the classical Black-Scholes theory by the quantum mechanical process $j_t(X) = U_t^* X \otimes 1 U_t$ where, for each $t \geq 0$, U_t is a unitary operator defined on the tensor product $\mathcal{H} \otimes \Gamma(L^2(\mathbf{R}_+, \mathcal{C}))$ of a system Hilbert space \mathcal{H} and the noise Boson Fock space $\Gamma = \Gamma(L^2(\mathbf{R}_+, \mathcal{C}))$ satisfying

$$(2.1) \quad dU_t = - \left(\left(iH + \frac{1}{2} L^* L \right) dt + L^* S dA_t - L dA_t^\dagger + (1 - S) d\Lambda_t \right) U_t, \quad U_0 = 1$$

where $X > 0$, H, L, S are in $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on \mathcal{H} , with S unitary and X, H self-adjoint. We identify time-independent, bounded, system space operators x with their ampliation $x \otimes 1$ to $\mathcal{H} \otimes \Gamma(L^2(\mathbf{R}_+, \mathcal{C}))$. The value process V_t is defined for $t \in [0, T]$ by $V_t = a_t j_t(X) + b_t \beta_t$ with terminal condition $V_T = (j_T(X) - K)^+ = \max(0, j_T(X) - K)$, where $K > 0$ is a bounded self-adjoint system operator corresponding to the strike price of the quantum option, a_t is a real-valued function, b_t is in general an observable quantum stochastic processes (i.e b_t is a self-adjoint operator for each $t \geq 0$) and $\beta_t = \beta_0 e^{tr}$ where β_0 and r are positive real numbers. Therefore $b_t = (V_t - a_t j_t(X)) \beta_t^{-1}$. We interpret the above in the sense of expectation i.e given $u \otimes \psi(f)$ in the exponential domain of $\mathcal{H} \otimes \Gamma$, where we will always assume $u \neq 0$ so that $\|u \otimes \psi(f)\| \neq 0$,

$$\begin{aligned} \langle u \otimes \psi(f), V_t u \otimes \psi(f) \rangle &= a_t \langle u \otimes \psi(f), j_t(X) u \otimes \psi(f) \rangle \\ &+ \langle u \otimes \psi(f), b_t u \otimes \psi(f) \rangle \beta_t \end{aligned}$$

(i.e the value process is always in reference to a particular quantum mechanical state, so we can eventually reduce to real numbers) and

$$\begin{aligned} \langle u \otimes \psi(f), V_T u \otimes \psi(f) \rangle &= \langle u \otimes \psi(f), (j_T(X) - K)^+ u \otimes \psi(f) \rangle \\ &= \max(0, \langle u \otimes \psi(f), (j_T(X) - K) u \otimes \psi(f) \rangle). \end{aligned}$$

As in the classical case we assume that the portfolio $(a_t, b_t), t \in [0, T]$ is self-financing i.e

$$dV_t = a_t dj_t(X) + b_t d\beta_t$$

or equivalently

$$da_t \cdot j_t(X) + da_t \cdot dj_t(X) + db_t \cdot \beta_t + db_t \cdot d\beta_t = 0.$$

Remark 1.

The fact that the value process (and all other operator processes X_t appearing in this paper) is always in reference to a particular quantum mechanical state, allows for a direct translation of all classical financial concepts described in Section 1 to the quantum case by considering the expectation (or matrix element) $\langle u \otimes \psi(f), X_t u \otimes \psi(f) \rangle$ of the process at each time t . If the process is classical (i.e, if $X_t \in \mathbb{R}$) then we may divide out $\|u \otimes \psi(f)\|^2$ and everything is reduced to the classical case described in Section 1.

Lemma 1. *Let $j_t(X) = U_t^* X \otimes 1 U_t$ where $\{U_t / t \geq 0\}$ is the solution of (2.1). If*

$$\alpha = [L^*, X] S, \quad \alpha^\dagger = S^* [X, L], \quad \lambda = S^* X S - X,$$

and

$$\theta = i[H, X] - \frac{1}{2} \{L^* L X + X L^* L - 2 L^* X L\}$$

then

$$(2.2) \quad dj_t(X) = j_t(\alpha^\dagger) dA_t^\dagger + j_t(\lambda) d\Lambda_t + j_t(\alpha) dA_t + j_t(\theta) dt$$

and for $k \geq 2$

$$(2.3) \quad (dj_t(X))^k = j_t(\lambda^{k-1} \alpha^\dagger) dA_t^\dagger + j_t(\lambda^k) d\Lambda_t + j_t(\alpha \lambda^{k-1}) dA_t + j_t(\alpha \lambda^{k-2} \alpha^\dagger) dt$$

Proof. Equation (2.2) is a standard result of quantum flows theory (cf. [13]). To prove (2.3) we notice that for $k = 2$, using (2.2), the Itô table

\cdot	dA_t^\dagger	$d\Lambda_t$	dA_t	dt
dA_t^\dagger	0	0	0	0
$d\Lambda_t$	dA_t^\dagger	$d\Lambda_t$	0	0
dA_t	dt	dA_t	0	0
dt	0	0	0	0

and the homomorphism property $j_t(xy) = j_t(x)j_t(y)$, we obtain

$$(dj_t(X))^2 = dj_t(X) dj_t(X) = j_t(\lambda \alpha^\dagger) dA_t^\dagger + j_t(\lambda^2) d\Lambda_t + j_t(\alpha \lambda) dA_t + j_t(\alpha \alpha^\dagger) dt$$

so (2.3) is true for $k = 2$. Assuming (2.3) to be true for k we have

$$\begin{aligned}
(dj_t(X))^{k+1} &= dj_t(X) (dj_t(X))^k \\
&= dj_t(X) \left(j_t(\lambda^{k-1} \alpha^\dagger) dA_t^\dagger + j_t(\lambda^k) d\Lambda_t + j_t(\alpha \lambda^{k-1}) dA_t + j_t(\alpha \lambda^{k-2} \alpha^\dagger) dt \right) \\
&= j_t(\lambda^k \alpha^\dagger) dA_t^\dagger + j_t(\lambda^{k+1}) d\Lambda_t + j_t(\alpha \lambda^k) dA_t + j_t(\alpha \lambda^{k-1} \alpha^\dagger) dt
\end{aligned}$$

Thus (2.3) is true for $k + 1$ also. □

3. DERIVATION OF THE QUANTUM BLACK-SCHOLES EQUATION

In the spirit of the previous section, let $V_t := F(t, j_t(X))$ where $F : [0, T] \times \mathcal{B}(\mathcal{H} \otimes \Gamma) \rightarrow \mathcal{B}(\mathcal{H} \otimes \Gamma)$ is the extension to self-adjoint operators $x = j_t(X)$ of the analytic function $F(t, x) = \sum_{n,k=0}^{+\infty} a_{n,k}(t_0, x_0) (t - t_0)^n (x - x_0)^k$, where x and $a_{n,k}(t_0, x_0)$ are in \mathbf{C} , and for $\lambda, \mu \in \{0, 1, \dots\}$

$$\begin{aligned}
F_{\lambda\mu}(t, x) &:= \frac{\partial^{\lambda+\mu} F}{\partial t^\lambda \partial x^\mu}(t, x) \\
&= \sum_{n=\lambda, k=\mu}^{+\infty} \frac{n!}{(n-\lambda)!} \frac{k!}{(k-\mu)!} a_{n,k}(t_0, x_0) (t - t_0)^{n-\lambda} (x - x_0)^{k-\mu}
\end{aligned}$$

and so, if 1 denotes the identity operator then

$$a_{n,k}(t_0, x_0) = a_{n,k}(t_0, x_0) 1 = \frac{1}{n! k!} F_{n,k}(t_0, x_0).$$

Notice that for $(t_0, x_0) = (0, 0)$ we have

$$V_t = \sum_{n,k=0}^{+\infty} a_{n,k}(0, 0) t^n j_t(X)^k = \sum_{n,k=0}^{+\infty} a_{n,k}(0, 0) t^n j_t(X^k).$$

Proposition 1. (*Quantum Black-Scholes Equation*)

$$\begin{aligned}
a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) j_t(\theta) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha \lambda^{k-2} \alpha^\dagger) = \\
a_t j_t(\theta) + V_t r - a_t j_t(X) r
\end{aligned}$$

(this is the quantum analogue of the classical Black-Scholes equation) and

$$\begin{aligned}
a_{0,1}(t, j_t(X)) j_t(\alpha^\dagger) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\lambda^{k-1} \alpha^\dagger) &= a_t j_t(\alpha^\dagger) \\
a_{0,1}(t, j_t(X)) j_t(\alpha) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha \lambda^{k-1}) &= a_t j_t(\alpha) \\
\sum_{k=1}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\lambda^k) &= a_t j_t(\lambda).
\end{aligned}$$

Proof. By Lemma 2.1 and the Itô table for quantum stochastic differentials

$$\begin{aligned}
dV_t &= dF(t, j_t(X)) = F(t+dt, j_{t+dt}(X)) - F(t, j_t(X)) \\
&= F(t+dt, j_t(X) + dj_t(X)) - F(t, j_t(X)) \\
&= \sum_{\substack{n,k=0 \\ n+k>0}}^{+\infty} a_{n,k}(t, j_t(X)) (dt)^n (dj_t(X))^k \\
&= a_{1,0}(t, j_t(X)) dt + \sum_{k=1}^{+\infty} a_{0,k}(t, j_t(X)) (dj_t(X))^k \\
&= a_{1,0}(t, j_t(X)) dt + a_{0,1}(t, j_t(X)) dj_t(X) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) \{j_t(\lambda^{k-1} \alpha^\dagger) dA_t^\dagger \\
&\quad + j_t(\lambda^k) d\Lambda_t + j_t(\alpha \lambda^{k-1}) dA_t + j_t(\alpha \lambda^{k-2} \alpha^\dagger) dt\}
\end{aligned}$$

where $\alpha, \alpha^\dagger, \lambda$ are as in Lemma 2.1. Thus

$$\begin{aligned}
dV_t &= \left(a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) j_t(\theta) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha \lambda^{k-2} \alpha^\dagger) \right) dt \\
&\quad + \left(a_{0,1}(t, j_t(X)) j_t(\alpha^\dagger) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\lambda^{k-1} \alpha^\dagger) \right) dA_t^\dagger \\
&\quad + \left(a_{0,1}(t, j_t(X)) j_t(\alpha) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha \lambda^{k-1}) \right) dA_t \\
(3.1) \quad &+ \sum_{k=1}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\lambda^k) d\Lambda_t
\end{aligned}$$

where θ is as in Lemma 2.1. We can obtain another expression for dV_t with the use of the self-financing property. We have

$$\begin{aligned}
dV_t &= a_t dj_t(X) + b_t d\beta_t = a_t dj_t(X) + b_t \beta_t r dt \\
&= a_t dj_t(X) + (V_t - a_t j_t(X)) \beta_t^{-1} \beta_t r dt \\
&= a_t dj_t(X) + (V_t - a_t j_t(X)) r dt \\
&= a_t \left(j_t(\alpha^\dagger) dA_t^\dagger + j_t(\lambda) d\Lambda_t + j_t(\alpha) dA_t + j_t(\theta) dt \right) + (V_t - a_t j_t(X)) r dt
\end{aligned}$$

which can be written as

$$\begin{aligned}
dV_t &= (a_t j_t(\theta) + V_t r - a_t j_t(X) r) dt + a_t j_t(\alpha^\dagger) dA_t^\dagger + a_t j_t(\alpha) dA_t \\
(3.2) \quad &+ a_t j_t(\lambda) d\Lambda_t
\end{aligned}$$

Equating the coefficients of dt and the quantum stochastic differentials in (3.1) and (3.2) we obtain the desired equations. \square

4. THE CASE $S = 1$: QUANTUM BROWNIAN MOTION

Proposition 2. *Let F be as in the previous section. If $S = 1$ then the equations of Proposition 3.1 combine into*

$$u_{10}(t, x) = \frac{1}{2} u_{02}(t, x) g(x) + u_{01}(t, x) h(x) - u(t, x) r$$

with initial condition $u(0, j_T(X)) = (j_T(X) - K)^+$ where $u(t, x) = F(T - t, x)$, $g(x) = [y^*, x][x, y]$, $h(x) = x r$ and $x, y \in \mathcal{B}(\mathcal{H} \otimes \Gamma)$

Proof. If $S = 1$ then, in the notation of Lemma 2.1, $\alpha = [L^*, X]$, $\alpha^\dagger = [X, L]$, $\lambda = 0$, and $\theta = i[H, X] - \frac{1}{2} \{L^* L X + X L^* L - 2 L^* X L\}$ and the equations of Proposition 3.1 reduce to

$$a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) j_t(\theta) + a_{0,2}(t, j_t(X)) j_t(\alpha \alpha^\dagger) = a_t j_t(\theta) + V_t r - a_t j_t(X) r$$

and

$$\begin{aligned}
a_{0,1}(t, j_t(X)) j_t(\alpha^\dagger) &= a_t j_t(\alpha^\dagger) \\
a_{0,1}(t, j_t(X)) j_t(\alpha) &= a_t j_t(\alpha)
\end{aligned}$$

which are condensed into

$$a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) j_t(\theta) + a_{0,2}(t, j_t(X)) j_t(\alpha \alpha^\dagger) = a_t j_t(\theta) + V_t r - a_t j_t(X) r$$

and

$$a_{0,1}(t, j_t(X)) = a_t.$$

Upon substituting the second of the last two equations into the first one and simplifying we obtain

$$a_{1,0}(t, j_t(X)) + a_{0,2}(t, j_t(X)) j_t([L^*, X] [X, L]) + a_{0,1}(t, j_t(X)) j_t(X) r - V_t r = 0$$

which can be written as

$$F_{10}(t, j_t(X)) + \frac{1}{2} F_{02}(t, j_t(X)) j_t([L^*, X] [X, L]) + F_{01}(t, j_t(X)) j_t(X) r = F(t, j_t(X)) r$$

with terminal condition $F(T, j_T(X)) = (j_T(X) - K)^+$. Letting $x = j_t(X)$, $y = j_t(L)$ be arbitrary elements in $\mathcal{B}(\mathcal{H} \otimes \Gamma)$ and letting $g(x) = [y^*, x] [x, y]$, $h(x) = x r$, we obtain

$$F_{10}(t, x) + \frac{1}{2} F_{02}(t, x) g(x) + F_{01}(t, x) h(x) = F(t, x) r.$$

Letting $u(t, x) := F(T - t, x)$, $u_{10}(t, x) = -F_{10}(T - t, x)$, $u_{02}(t, x) = F_{02}(T - t, x)$ and $u_{01}(t, x) = F_{01}(T - t, x)$ we obtain

$$u_{10}(t, x) = \frac{1}{2} u_{02}(t, x) g(x) + u_{01}(t, x) h(x) - u(t, x) r$$

with $u(0, j_T(X)) = (j_T(X) - K)^+$.

□

5. THE CASE $S \neq 1$: QUANTUM POISSON PROCESS

In this section we examine the equations of Proposition 3.1 under the assumption $S \neq 1$.

Proposition 3. *Let F be as in Section 3. If $[X, S] = S$ then the equations of Proposition 3.1 combine into*

$$u_{10}(t, x) = \sum_{k=2}^{+\infty} \frac{1}{k!} u_{0k}(t, x) g(x) + u_{01}(t, x) h(x) - u(t, x) r$$

with initial condition $u(0, j_T(X)) = (j_T(X) - K)^+$ where $u(t, x) = F(t - T, x)$, $g(x) = [y^*, x] [x, y] - i [z, x] + \frac{1}{2} \{y^* y x + x y^* y - 2 y^* x y\}$, $h(x) = x r$ and $x, y, z \in \mathcal{B}(\mathcal{H} \otimes \Gamma)$

Proof. Since X is self-adjoint and S is unitary, assuming that $[X, S] = S$ is equivalent to assuming that $\lambda = S^* X S - X = 1$ and the equations of Proposition 3.1 take the form

$$a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) j_t(\theta) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha \alpha^\dagger) = a_t j_t(\theta) + V_t r - a_t j_t(X) r$$

and

$$\begin{aligned}
a_{0,1}(t, j_t(X)) j_t(\alpha^\dagger) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha^\dagger) &= a_t j_t(\alpha^\dagger) \\
a_{0,1}(t, j_t(X)) j_t(\alpha) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha) &= a_t j_t(\alpha) \\
\sum_{k=1}^{+\infty} a_{0,k}(t, j_t(X)) &= a_t
\end{aligned}$$

which are satisfied if

$$a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) j_t(\theta) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha \alpha^\dagger) = a_t j_t(\theta) + V_t r - a_t j_t(X) r$$

and $a_t = \sum_{k=1}^{+\infty} a_{0,k}(t, j_t(X))$ which, if substituted in the previous one, yields

$$a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) j_t(X) r + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) (j_t(\alpha \alpha^\dagger - \theta) + j_t(X) r) = V_t r.$$

But

$$\begin{aligned}
j_t(\alpha \alpha^\dagger - \theta) &= [j_t(L)^*, j_t(X)] [j_t(X), j_t(L)] - i [j_t(H), j_t(X)] \\
&+ \frac{1}{2} \{j_t(L)^* j_t(L) j_t(X) + j_t(X) j_t(L)^* j_t(L) - 2 j_t(L)^* j_t(X) j_t(L)\}
\end{aligned}$$

Letting $x = j_t(X)$, $y = j_t(L)$, $z = j_t(H)$, $h(x) = x r$ and

$$g(x) = [y^*, x] [x, y] - i [z, x] + \frac{1}{2} \{y^* y x + x y^* y - 2 y^* x y\}$$

using the notation of the previous section we obtain the Black-Scholes equation for the case $S \neq 1$ as stated in the Proposition. □

6. SOLUTION OF THE QUANTUM BROWNIAN MOTION BLACK-SCHOLES EQUATION

To solve the Quantum Brownian motion Black-Scholes equation we assume that $j_t(X^2) = j_t([L^*, X] [X, L])$ which is the same as $X^2 = [L^*, X] [X, L]$. Since $X = X^*$, it follows that $[L^*, X] = [X, L]^*$ and so letting $\phi(X) = [X, L]$ we find $X^2 = \phi(X)^* \phi(X)$ i.e $\phi(X) = W X$ which implies that $[X, L] = W X$ and $[L^*, X] = X W^*$, where W is

an arbitrary unitary operator acting on the system space. In this case equation (2.2) takes the form

$$dj_t(X) = j_t \left(i [H, X] + \frac{1}{2} (L^* W X + X W^* L) \right) dt + j_t(X W) dA_t^\dagger + j_t(W^* X) dA_t.$$

Lemma 2. *If $H > 0$ is a bounded self-adjoint operator on a Hilbert space \mathcal{H} then there exists a bounded self-adjoint operator A on \mathcal{H} such that $H = e^A$.*

Proof. Let $H = \int_a^b \lambda dE_\lambda$ where $[a, b] \subset (0, +\infty)$ and $a \leq \|H\| \leq b$. Letting $\lambda = e^\mu$ we obtain $H = \int_{\ln a}^{\ln b} e^\mu dF(\mu)$ where $F(\mu) = E(e^\mu)$. Thus $H = e^A$ where $A = \int_{\ln a}^{\ln b} \mu dF(\mu)$ with $\|A\| \leq \max(|\ln a|, |\ln b|)$. To show that the family $\{F(\mu)/\ln a \leq \mu \leq \ln b\}$ is a resolution of the identity we notice that for $h \in \mathcal{H}$ and $\lambda, \mu \in [\ln a, \ln b]$ we have:

- (i) $F(\lambda) F(\mu) = E(e^\lambda) E(e^\mu) = E(e^\lambda \wedge e^\mu) = F(\lambda \wedge \mu),$
- (ii) $\lim_{\lambda \rightarrow \mu^-} F(\lambda) h = \lim_{e^\lambda \rightarrow e^\mu^-} E(e^\lambda) h = E(e^\mu) h = F(\mu) h,$
- (iii) $\lambda < \mu \Rightarrow e^\lambda < e^\mu \Rightarrow E(e^\lambda) < E(e^\mu) \Rightarrow F(\lambda) < F(\mu),$
- (iv) $\lambda < \ln a \Rightarrow e^\lambda < a \Rightarrow E(e^\lambda) = 0 \Rightarrow F(\lambda) = 0,$
- (v) $\lambda > \ln b \Rightarrow e^\lambda > b \Rightarrow E(e^\lambda) = 1 \Rightarrow F(\lambda) = 1.$

and the proof is complete. □

The equation in Proposition 4.1 now has the form

$$u_{10}(t, x) = \frac{1}{2} u_{02}(t, x) x^2 + u_{01}(t, x) x r - u(t, x) r$$

with initial condition $u(0, j_T(X)) = (j_T(X) - K)^+$ where we may assume that x is a bounded self-adjoint operator. Since

$$u(t, x) = F(T - t, x) = \sum_{n,k=0}^{+\infty} a_{n,k}(0, 0) (T - t)^n x^k$$

and $x = j_t(X) > 0$, and K are invertible, we may let $x = K e^z$ where z is a bounded self-adjoint operator commuting with K , and obtain

$$\begin{aligned}
\omega(t, z) &:= u(t, K e^z) = \sum_{n,k=0}^{+\infty} a_{n,k}(0, 0) (T-t)^n (K e^z)^k \\
\omega_{01}(t, z) &= \sum_{n,k=0}^{+\infty} a_{n,k}(0, 0) (T-t)^n k (K e^z)^k = \sum_{n=0,k=1}^{+\infty} a_{n,k}(0, 0) (T-t)^n k x^k \\
&= \sum_{n=0,k=1}^{+\infty} a_{n,k}(0, 0) (T-t)^n k x^{k-1} x = u_{01}(t, x) x
\end{aligned}$$

Similarly

$$\begin{aligned}
\omega_{02}(t, z) &= \sum_{n=0,k=1}^{+\infty} a_{n,k}(0, 0) (T-t)^n k^2 (K e^z)^k = \sum_{n=0,k=1}^{+\infty} a_{n,k}(0, 0) (T-t)^n k^2 x^k \\
&= \sum_{n=0,k=1}^{+\infty} a_{n,k}(0, 0) (T-t)^n (k(k-1) + k) x^k \\
&= \sum_{n=0,k=2}^{+\infty} a_{n,k}(0, 0) (T-t)^n k(k-1) x^{k-2} x^2 \\
&+ \sum_{n=0,k=1}^{+\infty} a_{n,k}(0, 0) (T-t)^n k x^{k-1} x \\
&= u_{02}(t, x) x^2 + u_{01}(t, x) x
\end{aligned}$$

and so

$$\omega_{02}(t, z) - \omega_{01}(t, z) = u_{02}(t, x) x^2.$$

Finally

$$\begin{aligned}
\omega_{10}(t, z) &= - \sum_{n=1,k=0}^{+\infty} a_{n,k}(0, 0) n (T-t)^{n-1} (K e^z)^k \\
&= - \sum_{n=1,k=0}^{+\infty} a_{n,k}(0, 0) n (T-t)^{n-1} x^k = u_{10}(t, x)
\end{aligned}$$

and so

$$(6.1) \quad \omega_{10}(t, z) = \frac{1}{2} \omega_{02}(t, z) + \omega_{01}(t, z) \left(r - \frac{1}{2}\right) - \omega(t, z) r$$

with initial condition $\omega(0, z_T) = (j_T(X) - K)^+$ where z_T is defined by $K e^{z_T} = j_T(X)$.

Theorem 1. *In analogy with the classical case presented in section 1, the solution of (6.1) is given by*

$$\omega(t, z) = K e^z \Phi(g(t, K e^z)) - K \Phi(h(t, K e^z)) e^{-r t}$$

where

$$\begin{aligned} g(t, K e^z) &= z t^{-1/2} + (r + 0.5) t^{1/2} \\ h(t, K e^z) &= z t^{-1/2} + (r - 0.5) t^{1/2}, \end{aligned}$$

and

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2}\pi} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!} \frac{x^{2n+1}}{2n+1}$$

Proof. We have

$$\omega_{10}(t, z) = K e^z (\Phi \circ g)_{10}(t, K e^z) - K (\Phi \circ h)_{10}(t, K e^z) e^{-r t} + K (\Phi \circ h)(t, K e^z) r e^{-r t},$$

$$\omega_{01}(t, z) = K e^z (\Phi \circ g)(t, K e^z) + K e^z (\Phi \circ g)_{01}(t, K e^z) - K (\Phi \circ h)_{01}(t, K e^z) e^{-r t},$$

and

$$\begin{aligned} \omega_{02}(t, z) &= K e^z (\Phi \circ g)(t, K e^z) + 2 K e^z (\Phi \circ g)_{01}(t, K e^z) + K (\Phi \circ g)_{02}(t, K e^z) \\ &\quad - K (\Phi \circ h)_{02}(t, K e^z) e^{-r t} \end{aligned}$$

where

$$\begin{aligned} (\Phi \circ h)(t, K e^z) &= \frac{1}{2} + \frac{1}{\sqrt{2}\pi} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!} \frac{(z t^{-1/2} + (r - 0.5) t^{1/2})^{2n+1}}{2n+1} \\ (\Phi \circ g)(t, K e^z) &= \frac{1}{2} + \frac{1}{\sqrt{2}\pi} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!} \frac{(z t^{-1/2} + (r + 0.5) t^{1/2})^{2n+1}}{2n+1} \end{aligned}$$

Thus

$$\omega_{10}(t, z) - \frac{1}{2} \omega_{02}(t, z) - \omega_{01}(t, z) (r - \frac{1}{2}) + \omega(t, z) r = K (A e^{-r t} + e^z B)$$

where

$$A = -(\Phi \circ h)_{10}(t, K e^z) + \frac{1}{2}(\Phi \circ h)_{02}(t, K e^z) + (\Phi \circ h)_{01}(t, K e^z) \left(r - \frac{1}{2}\right)$$

$$B = (\Phi \circ g)_{10}(t, K e^z) - \frac{1}{2}(\Phi \circ g)_{02}(t, K e^z) - (\Phi \circ g)_{01}(t, K e^z) \left(r + \frac{1}{2}\right)$$

It follows that $A = B = 0$ thus proving (6.1). Moreover, in order to prove that the initial condition is satisfied, we have

$$\begin{aligned} \omega(0, z_T) &= K e^{z_T} \Phi(g(0, K e^{z_T})) - K \Phi(h(0, K e^{z_T})) \\ &= (K e^{z_T} - K) \Phi(g(0, K e^{z_T})) + K (\Phi(g(0, K e^{z_T})) - \Phi(h(0, K e^{z_T}))). \end{aligned}$$

But

$$g(0, K e^{z_T}) - h(0, K e^{z_T}) = \lim_{t \rightarrow 0^+} \left(\frac{z}{\sqrt{t}} + (r + 0.5) \sqrt{t} - \frac{z}{\sqrt{t}} - (r - 0.5) \sqrt{t} \right) = 0$$

and so $\Phi(g(0, K e^{z_T})) - \Phi(h(0, K e^{z_T})) = 0$. Thus, it suffices to show that

$$\Phi(g(0, K e^{z_T})) = \begin{cases} 1 & \text{if } K e^{z_T} \geq K \\ 0 & \text{if } K e^{z_T} < K \end{cases}$$

We have

$$\begin{aligned} \Phi(g(0, K e^{z_T})) &= \lim_{t \rightarrow 0^+} (\Phi \circ g)(t, K e^{z_T}) \\ &= \frac{1}{2} + \lim_{t \rightarrow 0^+} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!} \frac{1}{t^{n+1/2}} \frac{z_T^{2n+1}}{2n+1} \end{aligned}$$

Suppose that $K e^{z_T} \geq K$. Then $z_T \geq 0$ and by the spectral resolution theorem $z_T^{2n+1} = \int_0^{+\infty} \lambda^{2n+1} dE_\lambda$. So

$$\begin{aligned} \Phi(g(0, K e^{z_T})) &= \frac{1}{2} + \lim_{t \rightarrow 0^+} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!} \frac{1}{t^{n+1/2}} \int_0^{+\infty} \frac{\lambda^{2n+1}}{2n+1} dE_\lambda \\ &= \frac{1}{2} + \lim_{t \rightarrow 0^+} \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \int_0^{\frac{\lambda}{\sqrt{t}}} e^{-\frac{s^2}{2}} ds dE_\lambda \\ &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \int_0^{+\infty} e^{-\frac{s^2}{2}} ds dE_\lambda \\ &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \frac{\sqrt{2\pi}}{2} dE_\lambda = 1 \end{aligned}$$

Similarly, if $K e^{z_T} < K$ then $z_T < 0$ and if we let $z_T = -w_T$ where $w_T = \int_0^{+\infty} \lambda dE_\lambda > 0$, then

$$z_T^{2n+1} = (-1)^{2n+1} \int_0^{+\infty} \lambda^{2n+1} dE_\lambda = - \int_0^{+\infty} \lambda^{2n+1} dE_\lambda$$

and so, as before, $\Phi(g(0, K e^{z_T})) = \frac{1}{2} - \frac{1}{2} \cdot 1 = 0$. □

Corollary 1. *The reasonable price for a quantum option is $\omega(T, z_0)$ where ω is as in Theorem 6.1 and z_0 is defined by $X = K e^{z_0}$. The associated quantum portfolio (a_t, b_t) is given by*

$$\begin{aligned} a_t &= \omega_{01}(t - T, z_t) \\ b_t &= (\omega(T - t, z_t) - a_t j_t(X)) e^{-tr} \beta_0^{-1} \end{aligned}$$

where z_t is defined by $j_t(X) = K e^{z_t}$. (As in the classical case described in Section 1, a reasonable price is defined as one which when invested at time $t = 0$ in a mixed portfolio, allows the investor through a self-financing strategy to end up at time $t = T$ with an amount of

$$\begin{aligned} \langle u \otimes \psi(f), V_T u \otimes \psi(f) \rangle &= \langle u \otimes \psi(f), (j_T(X) - K)^+ u \otimes \psi(f) \rangle \\ &= \max(0, \langle u \otimes \psi(f), (j_T(X) - K) u \otimes \psi(f) \rangle) \end{aligned}$$

which is the same as the payoff, had the option been purchased. Here, $u \otimes \psi(f)$ is any vector in the exponential domain of $\mathcal{H} \otimes \Gamma$).

Proof. By Theorem 6.1, the reasonable price for a quantum option is $V_0 = F(0, j_0(X)) = F(0, X) = u(T, X) = \omega(T, z_0)$. The formulas for a_t and b_t follow from the definition of the portfolio, given in Section 2. □

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